Integrals Involving Hypergeometric Function of Four Variables

Samta Gulia*, Harish Singh**

***Abstract:** In this research paper we explain the first main case of Integrals Involving Hypergeometric Function of Four Variables of positive definite matrix of order m×m which has many other similar sub cases, a comprehensive list of these integrals is given. Proof of all integrals are similar, therefore, detailed proof is given in the case of integral (1) and so rest are quoted directly as below.*

***Keywords:** Hypergeometric function, binomial, exponential, logarithmic and trigonometrical series*

1. INTRODUCTION

First to explain this paper, we should know about sequence and series as well as matrix sequence and matrix series. A matrix series is calculated by summation of all elements of matrix sequence. Let A_0, A_1, A_2, ………., A_n be the elements of matrix sequence then $M(A)=\sum_{m=0}^{\infty} A_m$, where M(A) is a matrix series.

If the matrix series is a power series then we will be introduce Hypergeometric series as follows:-

The Power series

$$1+\frac{a.b}{c}z+\frac{a(a+1)b.(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)b.(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^2}{3!}+\cdots\dots\dots\dots\dots.(1.1.1)$$

is one of the great generality; it includes as particular cases, most of the familiar series like the binomial, exponential, logarithmic and trigonometrical series.

For a=1 and b=c, the series reduces to simple geometrical series. From the fact that the series (1.1.1) is a generalization of the geometric series, it is called the Hypergeometric series. We apply a formula in our result by using Factorial function as we are familiar with the factorial of a positive integer. We write it as

$$n!=n(n-1)(n-2)\dots\dots\dots\dots\dots\dots\dots 3.2.1$$

We now define its generalization $(\propto)_n$ (read as $\propto$ suffix n) by the equations

$$(\propto)_n=\propto(\propto+1)(\propto+2)\dots\dots\dots\dots\dots\dots\dots(\propto+n-1), n\geq 1,$$

$$=\prod_{m=1}^{n}(\alpha+n-1)_n \; and \; (\alpha)_0=1, \alpha\neq 0.$$

This function is $(\propto)_n$ called the Factorial function. Clearly $(1)_n = n!$

For this factorial function we shall prove the result

$$(\propto)_{mn}= m^{nm}\left(\frac{\propto}{m}\right)_n\left(\frac{\propto+1}{m}\right)_n\dots\dots\dots\dots\dots\dots\left(\frac{\propto+m-1}{m}\right)_n,$$

Where m is a positive integer and n is a non-negative integer. By using above equations, We have the following formula

$$(\propto)_{mn}=m^{nm}\prod_{k=1}^{m}\left(\frac{\propto+k-1}{m}\right)_n \qquad (1.1.2)$$

We can define a general Hypergeometric series in real scalar variable Z as below

$${}_mF_n(\alpha_1, \alpha_2\dots\dots\dots \alpha_m; \beta_1, \beta_2\dots\dots\dots \beta_n; Z) = \sum_{r=0}^{\infty}\frac{(\alpha_1)_r..(\alpha_2)_r.......(\alpha_m)_r\, Z^r}{(\beta_1)_r.(\beta_2)_r.......(\beta_n)_r\, r!}$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\dots\dots\dots\dots(\alpha+m-1)$. Here α_1, $\alpha_2\dots\dots\alpha_m$ scalars and m upper parameters and β_1, $\beta_2\dots\dots\beta_n$ are scalars and n lower parameters. Above expression is defined for two, three and four variables for more see Exton[2, 3]. Initially nineteen Hypergeometric functions of four variables were introduced by Exton[4], then sixty four new Hypergeometric functions was formed. Further all forty integrals of four variables has generalized by Vyas et al[11]. To

*Assistant Professor, Government College, Badli, Haryana, India, samgulia@yahoo.in
**Professor, Maharaja Surajmal Insitute (GGSIPU), New Delhi, harishsingh@msi-ggsip.org

know more about this see Singh et al[9] and Chandel et al[8]. In this chapter we will discuss p.d.f of forty integrals.

In this paper we explained first case of Integral involving Hypergeometric Function of four variable of positive definite matrix of order m×m which has many other similar cases, a comprehensive list of these integrals is given. Proof of all integrals are similar therefore detailed proof is given in the case of integral (1) and so rest are quoted directly as below.

We shall take p, q, r and s to be positive integers of the symbols $\Delta(n, a)$ and X_r stand for the sequence of parameters $\frac{\alpha}{n}, \frac{\alpha+1}{n}, ..., \frac{\alpha+n-1}{n}$ and $W_1, W_2, ... W_r$ respectively.

2. MULTIVARIABLE FUNCTIONS

The great success of the theory of hypergeometric functions of a single variable has stimulated the development of a corresponding study of the theory in the two and more variables, in 1880, Appell introduced and studied systematically the four functions F_1, F_2, F_3 and F_4 which are the generalizations of the Gaussian hypergeometric functions of two variables. In the year Horn[5] defined ten hypergeometric function of two variables and denoted them by G_1, G_2, G_3, The confluent forms of the four Appell functions were studied by Humbert[6]. A list of these functions is given in the work by Erdely[1].

In a similar manner to the generalization of the single hypergeometric function the functions F_1 to F_4 and their confluent forms were further generalized by kampe de Feriet[7] who introduced a general hypergeometric function of two variables. The Kampe de Feriet function has further been generalized by Srivastana et al[10]. The explanation of Integrals of four variables was introduced by Singh et al[9]. under Statistical Distribution associated with Hypergeometric function of matrix arguments.

3. THE QUADRUPLE HYPERGEOMETRIC FUNCTIONS

No specific study has been made for any hypergeometric series of four variables, apart from the four variables Lauricella functions $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$and $F_D^{(4)}$ and certain for their limiting cases, and the extremely generalized multiple hypergeometric functions of Srivastava et al[10]. On account of the large number of such functions which arise from a systematic study of all possibilities, Exton restricted here to those functions which are complete and of order second and which involves at least one product of type $(a)_{p+q+r+s}$ in the series representation : p, q, r and s are the indices of quadruple summations.

The notations used for quadruple hypergeometric series are K_1, K_2 ..... ……

K_1 $(a_1, a_1, a_1, a_2 ; b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; ; x, y, z, t)$

$$\sum \frac{(a_1,p+q+r)(a_2,s)}{(c_1,p)(c_2,q)(c_3,r)(c_4,s)} x^p \,.\, y^q \,.\, z^r \,.\, t^s \tag{1.3.1}$$

Continuously like this we can make $k_2, k_3, k_4,$ ……….

It is presumed that parameters and arguments in above functions are so restricted that the series involved is convergent. Other quadruple hypergeometric series have been given by Sharma and Parihar. They introduced eighty three new hypergeometric series of variable four with the symbols $F_1^{(4)}, F_2^{(4)}...., F_{83}^{(4)}$ to quote we have.

F_1^4 $(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; x, y, z, t)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s} x^p y^q z^r t^s \tag{1.3.2}$$

Continuously like this we can make F_2^4 , F_3^4 , F_4^4 ……. Out of these eighty-three functions nineteen functions had already been introduced by Exton.

The recent work on quadruple hypergeometric series has been done by Chandel et al[8], they introduced seven more possible hypergeometric functions of four variables and thus completed the set of all possible quadruple hypergeometric series.

4. INTEGRALS OF MATRIX VARIATE HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

[1] $\int_0^1 |u|^{\alpha-1} |1-u|^{\beta-1} F_1(\alpha, 1-\alpha ; \gamma; \frac{u}{2})$

F_1^4 $[a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]du$

$$= B^1 F_1^4 \left[\begin{matrix} a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4 ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n, \gamma+\beta), \quad \Delta\left(\frac{n}{2}, \gamma+\beta+\alpha\right), \quad \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}\right); \quad w\ x\ y\ z \end{matrix} \right]$$

(1.4.1)

Where

F_1^4 $(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; w, x, y, z)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s} w^p x^q y^r z^s$$

so that

F_1^4 [a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4;|1-u|^n w, |1-u|^n x, |1-u|^n y, |1-u|^n z]

F_1^4 [|1-u|^n w, |1-u|^n x, |1-u|^n y, |1-u|^{n}z]

Then a probability density function (p.d.f.) of (1.4.1) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1} F_1(\chi_1)F_1^4(\chi_2)}{B^1 F_1^4(\chi_3)}$$

= 0 else where

Where

$$\chi_1 = (\alpha, 1\text{-}\alpha, \gamma; \frac{u}{2})$$

$$\chi_2 = [|1\text{-}u|^n w, |1\text{-}u|^n x, |1\text{-}u|^n y, |1\text{-}u|^n z]$$

$$\chi_3 = \begin{bmatrix} a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{bmatrix}$$

Proof:-

The L.H.S. in the integrand of the (1.4.1) is the expressing of the quadruple Hypergeometric function in terms of equivalent series. We find that the integral becomes

$$\int_0^I |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$$

$$F_1^4\left[a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; |1-u|^n w, |1-u|^n x, |1-u|^n y, |1-u|^n z\right]du$$

Above expression can be written as

$$\int_0^I |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$$

$$\sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s}\left[|1-u|^n w\right]^p\left[|1-u|^n x\right]^q\left[|1-u|^n y\right]^r\left[|1-u|^n z\right]^s$$

(1.4.1.1)

We assume that the series is uniformly convergent in the region of integration, the inversion of integration and summation is infinite, then integral

$$= \int_0^I |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$$

$$\times \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q}\left[w^p|1-u|^{pn} x^q|1-u|^{qn} y^r|1-u|^{rm} z^s |1-u|^{sn}\right]du$$

(1.4.1.2)

$$\text{Where } A_{r,s}^{p,q} = \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s}$$

$$= \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q} w^p x^q y^r z^s \int_0^I |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2}) . \left[|1-u|^{n(p+q+r+s)}\right]du$$

$$= \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q} w^p x^q y^r z^s$$

$$\int_0^I |u|^{\alpha-1} |1-u|^{\beta+n(p+q+r+s)-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})du$$

(1.4.1.3)

By using the below formula on evaluating the integral by means of the formula

$$\int_0^I |x|^{\alpha-1}|1-x|^{\beta-1} F_1(\alpha, 1-\alpha ; \gamma; \frac{u}{2})dx$$

$$= \frac{\Gamma_m(\gamma)\Gamma_m(\beta)\Gamma_m(\frac{\gamma+\beta}{2})\Gamma_m(\frac{1+\gamma+\beta}{2})}{\Gamma_m(\gamma+\beta)\Gamma_m\left[\frac{(\gamma+\beta+\alpha)}{2}\right]\Gamma_m\left[\frac{(1+\gamma+\beta-\alpha)}{2}\right]} \quad (1.4.1.4)$$

Where Re(α), Re (β) > 0
We see that the value of the integral (1.4.1.3) becomes by using (1.4.1.4)

$$= \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q} w^p x^q y^r z^s$$

$$\times \frac{\Gamma_m(\gamma)\,\Gamma_m(\beta+n(p+q+r+s))\,\Gamma_m\left(\frac{\gamma+\beta+n(p+q+r+s)}{2}\right)}{\Gamma_m(\gamma+\beta+n(p+q+r+s))\,\Gamma_m\left[\frac{(\gamma+\beta+n(p+q+r+s)+\alpha}{2}\right]}$$

$$\times \frac{\Gamma_m\left(\frac{1+\gamma+\beta+n(p+q+r+s)}{2}\right)}{\Gamma_m\left[\frac{(1+\gamma+\beta+n(p+q+r+s)-\alpha}{2}\right]}$$

$$= \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q} w^p x^q y^r z^s$$

$$\times \frac{\Gamma_m(\gamma)\,\Gamma_m[(\beta)+n(p+q+r+s)]\ \Gamma_m\left[\left(\frac{\gamma+\beta}{2}\right)+\left(\frac{n(p+q+r+s)}{2}\right)\right]}{\Gamma_m\,[(\gamma+\beta)+n(p+q+r+s)]\ \Gamma_m\left[\frac{\gamma+\beta+\alpha}{2}+\frac{n(p+q+r+s)}{2}\right]}$$

$$\times \frac{\Gamma_m\left[\frac{(1+\gamma+\beta)}{2}+\frac{n(p+q+r+s)}{2}\right]\left[\left(\frac{(1+\gamma+\beta)}{2}\right)\right]}{\Gamma_m\left[\frac{(1+\gamma+\beta-\alpha)}{2}+\frac{n(p+q+r+s)}{2}\right]}$$

(1.4.1.5)

Using the formula;

$$(a)_r = \Gamma(a+r) \big/ \Gamma(a) \Rightarrow \Gamma(a+r) = \Gamma(a).(a)_r$$

(1.4.1.6)

Equation (1.4.1.5) become by using above formula (1.4.1.6)

$$= \sum_{p,q,r,s=0}^{\infty} A_{r,s}^{p,q} w^p x^q y^r z^s$$

$$\frac{\Gamma_m(\gamma)\Gamma_m(\beta)\Gamma_m(\frac{\gamma+\beta}{2})\Gamma_m(\frac{1+\gamma+\beta}{2})}{\Gamma_m(\gamma+\beta)\Gamma_m\left[\frac{(\gamma+\beta+\alpha)}{2}\right]\Gamma_m\left[\frac{(1+\gamma+\beta-\alpha)}{2}\right]} \times$$

$$\times \frac{(\beta)_{n(p+q+r+s)}\left(\frac{\gamma+\beta}{2}\right)_{\frac{n}{2}(p+q+r+s)}\left(\frac{1+\gamma+\beta}{2}\right)_{\frac{n}{2}(p+q+r+s)}}{(\gamma+\beta)_{n(p+q+r+s)}\left[\frac{(\gamma+\beta+\alpha)}{2}\right]_{\frac{n}{2}(p+q+r+s)}\left[\frac{(1+\gamma+\beta-\alpha)}{2}\right]_{\frac{n}{2}(p+q+r+s)}}$$

$$= B' \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s} \times$$

$$\times \frac{(\beta)_{n(p+q+r+s)}\left(\frac{\gamma+\beta}{2}\right)_{\frac{n}{2}(p+q+r+s)}\left(\frac{1+\gamma+\beta}{2}\right)_{\frac{n}{2}(p+q+r+s)}}{(\gamma+\beta)_{n(p+q+r+s)}\left[\frac{(\gamma+\beta+\alpha)}{2}\right]_{\frac{n}{2}(p+q+r+s)}\left[\frac{(1+\gamma+\beta-\alpha)}{2}\right]_{\frac{n}{2}(p+q+r+s)}} w^p x^q y^r z^s$$

Where B^1=

$$B\frac{\Gamma_m(\frac{\gamma+\beta}{2})\Gamma_m(\frac{1+\gamma+\beta}{2})}{\Gamma_m\left[\frac{(\gamma+\beta+\alpha)}{2}\right]\Gamma_m\left[\frac{(1+\gamma+\beta-\alpha)}{2}\right]}, B = \Gamma_m(\gamma)\Gamma_m(\beta) \big/ \Gamma_m(\gamma+\beta)$$

Where B is a Beta Function.

Now, if we apply the Formula as describes as (1.1.2)

$$(a)_{k1} = k^{k1}\prod_{j=1}^{k}\left\{\frac{(a+j-1)}{k}\right\}$$

Where k is a positive integral and non negative, there after little simplification we arrive

at the result [1] is

$$= B^1 F_1^4\left[\begin{matrix} a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; \Delta(n,\beta), \Delta(\frac{n}{2},\frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2},\frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta),\quad \Delta\left(\frac{n}{2},\frac{\gamma+\beta+\alpha}{2}\right),\quad \Delta\left(\frac{n}{2},\frac{1+\gamma+\beta-\alpha}{2}\right);\quad w\ x\ y\ z \end{matrix}\right]$$

(1.4.1.7)

[2] $\int_0^1 \quad |u|^{\alpha-1} |1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$

$F_2^4[w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]$ du

$$= B^1 F_2^4\left[\begin{matrix} a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; \Delta n,\beta), \Delta(\frac{n}{2},\frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2},\frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta),\quad \Delta\left(\frac{n}{2},\frac{\gamma+\beta+\alpha}{2}\right),\quad \Delta\left(\frac{n}{2},\frac{1+\gamma+\beta-\alpha}{2}\right);\quad w\ x\ y\ z \end{matrix}\right] \quad (1.4.2)$$

Where $F_2^4[a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; (w, x, y, z)]$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q}(a_2)_{r+s}(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s} w^p x^q y^r z^s$$

Then a probability density function (p.d.f.) of (1.4.2) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1}\, F_1(\chi_1)F_2^4(\chi_2)}{B^1 F_2^4(\chi_4)}$$

= 0 else where

Where

$$\chi_4 = \left[\begin{array}{l} a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{array}\right]$$

[3] $\int_0^1 \; |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$

$F_3^4\,[w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]\, du$

$$= B^1 F_3^4 \left[\begin{array}{l} a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; (n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \quad \Delta\left(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}\right), \quad \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}\right); \quad w\ x\ y\ z \end{array}\right]$$

(1.4.3)

Where

$F_3^4\,(a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; w, x, y, z)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q}(a_2)_r(a_3)_s(b_1)_{p+q+r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s}\, w^p x^q y^r z^s$$

Then a probability density function (p.d.f.) of (1.4.3) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1}\, F_1(\chi_1)F_3^4(\chi_2)}{B^1 F_3^4(\chi_5)}$$

= 0 else where

$$\chi_5 = \left[\begin{array}{l} a_1, a_1, a_2, a_2, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4, ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{array}\right]$$

[4] $\int_0^1 \; |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$

$F_4^4\,[w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]\, du$

$$= B^1 F_4^4 \left[\begin{array}{l} a_1, a_1, a_1, a_1, b_1, b_1, b_1, b_1, c_1, c_2, c_3, c_4; (n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \quad \Delta\left(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}\right), \quad \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}\right); \quad w\ x\ y\ z \end{array}\right]$$

(1.4.4)

Where

$F_4^4\,(a_1, a_1, a_1, a_1, b_1, b_1, b_1, b_2, c_1, c_2, c_3, c_4, ; w, x, y, z)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r+s}(b_1)_{p+q+s}(b_2)_r}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s}\, w^p x^q y^r z^s$$

Then a probability density function (p.d.f.) of (1.4.4) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1}\, F_1(\chi_1)F_4^4(\chi_2)}{B^1 F_4^4(\chi_6)}$$

= 0 else where

Where

$$\chi_6 = \left[\begin{array}{l} a_1, a_1, a_1, a_1, b_1, b_1, b_1, b_2, c_1, c_2, c_3, c_4, ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{array}\right]$$

[5] $\int_0^1 \; |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$

$F_5^4\,[w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]\, du$

$$= B^1 F_5^4 \left[\begin{array}{l} a_1, a_1, a_1, a_2, b_1, b_1, b_2, b_2, c_1, c_2, c_3, c_4; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta((\gamma+\beta), \quad \Delta\left(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}\right), \quad \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}\right); \quad w\ x\ y\ z \end{array}\right]$$

(1.4.5)

Where

$F_5^4\,(a_1, a_1, a_1, a_2, b_1, b_1, b_2, b_2, c_1, c_2, c_3, c_4, ; w, x, y, z)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r+s}(a_2)_s(b_1)_{p+q}(b_2)_{r+s}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s}\, w^p x^q y^r z^s$$

Then a probability density function (p.d.f.) of (1.4.5) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1}\, F_1(\chi_1)F_5^4(\chi_2)}{B^1 F_5^4(\chi_7)}$$

= 0 else where

Where

$$\chi_7 = \left[\begin{array}{l} a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_2, c_1, c_2, c_3, c_4, ; \Delta(n,\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n,\gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{array}\right]$$

[6] $\int_0^1 |u|^{\alpha-1}|1-u|^{\beta-1} F_1(\alpha, 1-\alpha; \gamma; \frac{u}{2})$

$F_6^4[w|1-u|^n, x|1-u|^n, y|1-u|^n, z|1-u|^n]\,du$

$$= B^1 F_6^4 \left[\begin{matrix} a_1, a_1, a_1, a_2, b_1, b_2, b_3, b_1, c_1, c_2, c_3, c_4; \Delta(n, \beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(, \gamma+\beta), \quad \Delta\left(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}\right), \quad \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}\right); \qquad w\ x\ y\ z \end{matrix} \right]$$

(1.4.6)

Where

$F_6^4(a_1, a_1, a_1, a_2, b_1, b_2, b_3, b_1, c_1, c_2, c_3, c_4, ; w, x, y, z)$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a_1)_{p+q+r}(a_2)_s(b_1)_{p+s}(b_2)_{q+r}}{p!q!r!s!(c_1)_p(c_2)_q(c_3)_r(c_4)_s} w^p x^q y^r z^s$$

Then a probability density function of (1.4.6) is given by :

$$F(u) = \frac{|u|^{\alpha-1}|1-u|^{\beta-1} F_1(\chi_1) F_6^4(\chi_2)}{B^1 F_6^4(\chi_8)}$$

= 0 else where

Where

$$\chi_8 = \left[\begin{matrix} a_1, a_1, a_1, a_2, b_1, b_2, b_3, b_1, c_1, c_2, c_3, c_4, ; \Delta(n, \beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta}{2}), \Delta\left(\frac{n}{2}, \frac{1+\gamma+\beta}{2}\right) \\ \Delta(n, \gamma+\beta), \Delta(\frac{n}{2}, \frac{\gamma+\beta+\alpha}{2}), \Delta(\frac{n}{2}, \frac{1+\gamma+\beta-\alpha}{2}); w, x, y, z \end{matrix} \right]$$

Proof of rest of the integral from (1.4.2) to (1.4.6) are similar. Therefore direct results have quoted.

BIBLIOGRAPHY

[1] **Erdelyi JA.:** Transformation of the Hypergeometric functions of two variables prcc. Roy. Soc. of Edinburgh.1948; 378-385.

[2] **Exton H.:** On Certain confluent Hypergeometric functions of four variables. Ganita. 1973; 25(2): 23-24.

[3] **Exton H.:** Hypergeometric functions of three variables. J. Indian Acad. Math. 1982; 4 (2): 113-119.

[4] **Exton H.:** Multiple Hypergeometric function and applications. London: Ellis Horwood; 1976.

[5] **Horn, J.:** Hypergeometric Function Zweier Veranderlichen Math. Ann, 105 (1931), pp 381-407.

[6] **Humbert P.:** The confluent hypergeometric function of two variables, Proc. Roy, Soc. Edinburg. 1920-21; 41: 93-96.

[7] **Kample De Feriet, J.:** Les Functions Hypergeometriques d'order Superierur a deux variables C.R. and Sci. Paris 173 (1921), pp 401-404.

[8] **Chandel RCS, Agarwal RD, Kumar Hemant.** Hypergeometric functions of four variables and their Integral representation, the Mathematics Education J. 1992; Vol xxvi No. 2: 74-94.

[9] **Singh S, Deoram MS, Shankarsa PB.** Integrals of four variables with Statistical Distribution associated with Hypergeometric function of matrix arguments. Journal of Applied Research and Development Institute. 2013; 7 (4): 15-23.

[10] **Srivastava HM, Doaust MC.** Certain generalised Neumann expansion associated with Kampe de Feriet funcion. Nederl. akad. Wetencsch. Proc. Ser. A, 72-Indag. Nath. 1969: 449-457.

[11] **Vyas, Mahesh and Sethi P.L:** Four variable associated with hypergeometric functions, Ph.D Thesis, J.N.V. University Jodhpur (2001).